COHOMOLOGICAL EXPRESSION OF THE CURVATURE OF KÄHLER MODULI

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ABSTRACT. The Kähler cone of a compact Kähler manifold carries a natural Riemannian metric, given by the intersection product of its cohomology ring. We give cohomological expressions for the Levi-Civita connection and curvature tensor of this metric, and determine when the metric is complete.

INTRODUCTION

Let X be a compact Kähler manifold of complex dimension n. The Kähler cone of X is the set of Kähler classes, that is, $(1,1)$-classes that contain a Kähler metric. Each Kähler class defines an inner product on the space of $(1,1)$-classes and letting the classes vary defines a natural Riemannian metric on the Kähler cone.

This metric has been studied by Wilson [Wil04], Totaro [Tot04], Wilson and Trenner [TW11] and the author [Mag12], sometimes by embedding the Kähler cone into the space of smooth Hermitian metrics on the manifold via the Aubin–Calabi–Yau theorem, and sometimes by working inside the cohomology ring.

By working with smooth forms, one can compute the curvature tensor of this metric. Wilson [Wil04] made one such approach, using very interesting tools that the author hasn't seen deployed since. The author [Mag12] made another attempt, using the L^2 metric on the infinite-dimensional space of all Kähler metrics. Ultimately, the author cannot see that either approach descends again to the level of cohomology (at least in a way that makes the connection to the cohomology classes one started from obvious).

Those working in the cohomology ring have obtained more complete results. Wilson and Trenner [TW11] performed extensive computations in the cohomology of Calabi–Yau threefolds. Huybrechts [Huy01] focused his attention on the variation of primitive forms in the Kähler cone as the Kähler classes vary, which is very related to the metric we're interested in and its curvature. Totaro [Tot04] also considered this metric in the more general setting of Hessian metrics arising from homogeneous polynomials. What unites these efforts in the cohomology ring is their fearlessness in picking convenient local bases to work in. Wilson, Trenner and Totaro all end up

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with local coordinate expressions for the curvature tensor of the metric. Huybrechts does not, but he doesn't compute the curvature tensor, for his sights are set elsewhere.

The starting point of this paper was a slight feeling of dissatisfaction with these local coordinate expressions, as the author felt he didn't really understand what was going on; that he didn't see the forest for the trees. In our treatment we thus stay entirely within the cohomology ring of an arbitrary compact Kähler manifold, and avoid picking coordinates at all. This shifts the difficulty from either previous approach from dealing with smooth forms or complicated polynomials to computing various cup products in the cohomology ring. Those turn out to be pleasantly manageable.

The main novelty of this approach is that we obtain explicit and clear formulas for the Levi-Civita connection of the metric, and for its curvature tensor. For example, at a point ω with adjoint Lefschetz operator Λ, and for real $(1,1)$-classes u, v, z, w, the curvature tensor is

$$R(u,v,z,w) = -\tfrac{1}{4}\langle \Lambda(u \cup w), \Lambda(v \cup z)\rangle + \tfrac{1}{4}\langle \Lambda(u \cup z), \Lambda(v \cup w)\rangle.$$

However, that is where the good times end. We are unable to profit from these explicit formulas to improve on the previous general results on the curvature of the metric in any way. The main difficulty that frustrates us is plainly visible in the above formula: It involves the cup product in the cohomology ring of an arbitrary compact Kähler manifold, of which next to nothing can be said. To be able to compute the various derived tensors of the curvature tensor (the Ricci or scalar curvatures), or to estimate their magnitudes, one needs to be able to estimate the norm of the cup product of two classes in terms of the norms of the individual classes uniformly over the Kähler cone. As Wilson and Trenner show with an example, this is impossible in general; and the author knows of no conditions that one can impose on the manifolds under consideration that restrict the cohomology ring in suitable ways.

1. The Kähler cone

Let X be a compact Kähler manifold of dimension $\dim_{\mathbb{C}} X = n$.

Definition. The *Kähler cone* of X is the set

$$\mathcal{C}(X) = \{\omega \in \mathrm{H}^{1,1}(X,\mathbb{R}) \mid \omega \text{ contains a Kähler metric}\}.$$

If there can be no confusion about the underlying manifold X, we'll just write $\mathcal{C}$ for its Kähler cone. As the name suggests, this is an open cones in the finite-dimensional vector space $\mathrm{H}^{1,1}(X,\mathbb{R})$. The Kähler cone is the transcendental analogue of the ample cone of a projective variety. It is described by the transcendental version of the Nakai–Moishezon criteria due to Demailly and Paun [DP04]:

Theorem 1.1. *The Kähler cone of X is a connected component of the set of real $(1,1)$-cohomology classes that are numerically positive on analytic* *cycles, that is, classes* α *such that* $\int_Z a^p > 0$ *for every irreducible analytic set* Z *in* X *of dimension* p*.*

Since the Kähler cone $\mathcal{C}$ is an open set in a real vector space, we can view it as a smooth manifold in its own right. It is in fact naturally a Riemannian manifold, because each Kähler class defines an inner product on the Kähler cone via the Hodge star operator, and these inner products vary smoothly with the underlying class.

Let's agree on some notation before we find convenient expressions for this metric. If x is an element of the cohomology ring of X, we write $x^{[k]} := x^k/k!$ for all $k \geq 0$. This notation is quite convenient for calculations with Kähler forms in the cohomology ring of X; I learned it from Georg Schumacher.

Proposition 1.2. *Let* u, v *be elements of* $T_\omega\mathcal{C}$*. The Riemannian metric on* $\mathcal{C}$ *can also be defined as:*

(1) $$\langle u, v\rangle = \Lambda(u)\Lambda(v) - \Lambda^{[2]}(u \wedge v).$$

(2) *The quadratic form defined by the Hessian of* $-\log \mathrm{Vol}$*.*

Proof. That (1) agrees with the inner product that ω defines can be seen by taking the primitive decomposition of u and v, plugging it into (1) and calculating until the Hodge–Riemann bilinear relations say that we have the correct inner product.

Note that we can view ω as the tautological section $\mathcal{C} \to T_{\mathcal{C}}$ associated to the tangent bundle of any open set in a vector space. Then $d_v \mathrm{Vol} = -\Lambda(v)$ as we see by considering $\Lambda(v)\omega^{[n]} = v \cup \omega^{[n-1]}$. We then find that $\mathrm{Hess}(u, v) \mathrm{Vol} = d_u d_v \mathrm{Vol} = \langle u, v\rangle$ by comparing with (1). □

Remark. Let's write $\mathcal{C}_1 \subset \mathcal{C}$ for the set of volume-1 Kähler classes. It is a smooth submanifold of $\mathcal{C}$, and there is a Riemannian isometry

$$\mathbb{R} \times \mathcal{C}_1 \to \mathcal{C}, \quad (t, \omega) \mapsto e^{t/n}\omega,$$

where $\mathbb{R}$ has the Euclidean metric and $\mathcal{C}_1$ has the restriction of the metric on $\mathcal{C}$. As a submanifold of $\mathcal{C}$, the tangent space of $\mathcal{C}_1$ at ω is the space of ω-primitive classes.

Some authors have used this isometry to work on $\mathcal{C}_1$ when studying the metric on the Kähler cone, as anything interesting will obviously happen there. We will mostly leave this isometry alone and work in all of $\mathcal{C}$ instead, until the time comes to compute the curvature tensor, when we find ourselves unable to refuse the comforts of that subspace any longer.

The theorem of Demailly and Paun describes the boundary of the Kähler cone of a compact complex manifold. It consists of three parts:

(1) Limits of classes ω_t whose volume $\int_X \omega_t^{[n]}$ tends to zero.
(2) Limits of classes whose volume tends to infinity.
(3) Limits of classes whose volume tends to some positive real number, but there exists a proper irreducible complex subspace $Z \subset X$ of dimension $p \geq 1$ whose volume tends to zero.

Let us conspire to call $\mathcal{P} := \{\omega \in \mathrm{H}^{1,1}(X, \mathbb{R}) \mid \omega^{[n]} > 0\}$ the cone of volume classes on X. One of its connected components contains the Kähler cone, but is in almost all cases bigger than it.

Proposition 1.3. *The metric on the Kähler cone of X is complete if and only if the Kähler cone is a connected component of the volume cone.*

Proof. We first show that the classes on the first two parts of the boundary pose no problems. Let I be an interval in the real numbers and let $\gamma : I \to \mathcal{C}$ be a smooth path in $\mathcal{C}$ that approaches the boundary of $\mathcal{C}$. Let $I_m = [a_m, b_m]$ be an increasing exhaustion of I by compact intervals and let γ_m be the restriction of γ to I_m. Suppose that the volume $\mathrm{Vol}(X, \gamma_m)$ tends to either zero or infinity as m tends to infinity.

Lemma 1.4. *Let $I = [a, b]$ be a compact interval in the real numbers $\mathbb{R}$, and let $\gamma : I \to \mathcal{C}$ be a smooth path. The length of the path γ satisfies*

$$L(\gamma) \geq \frac{\sqrt{2}}{\sqrt{n}} \big|\log \mathrm{Vol}(X, \gamma(b)) - \log \mathrm{Vol}(X, \gamma(a))\big|.$$

Sketch of proof. We apply the Cauchy–Schwarz inequality to the scalar product $\langle u, \omega\rangle$; this gives

$$|d_u \log \mathrm{Vol}(X, \omega)|^2 = |\tfrac{1}{2}\langle u, \omega\rangle|^2 \leq \tfrac{n}{2}\langle u, u\rangle.$$

Integrating and applying the triangle inequality then gives the announced estimate. □

Applying the lemma on each interval I_m then gives that

$$L(\gamma) = \lim_{m \to +\infty} L(\gamma_m) = +\infty.$$

Thus the limit class $\lim \gamma(t)$ on the boundary cannot be approached by paths in $\mathcal{C}$ of finite length.

If the Kähler and volume cones of X do not coincide, then there exists a class α on the boundary of $\mathcal{C}$ such that $\mathrm{Vol}(X, \alpha) > 0$, but there is a proper complex subspace $Z \subset X$ such that $\mathrm{Vol}(Z, \alpha) = 0$.

As α is on the boundary of the Kähler cone, then there exists a Kähler class ω such that $\gamma(t) := \alpha + t\omega$ is in the Kähler cone for all $t > 0$. The tangent vectors of the path γ are $\gamma'(t) = \omega$, and the norm of $\gamma'(t)$ at the point $\gamma(t)$ is

$$h(t) := \langle \gamma'(t), \gamma'(t)\rangle(\gamma(t)) = \left(\frac{1}{\mathrm{Vol}(X, \gamma(t))} \int_X \omega \wedge (\alpha + t\omega)^{[n-1]}\right)^2 - \frac{1}{\mathrm{Vol}(X, \gamma(t))} \int_X \omega^2 \wedge (\alpha + t\omega)^{[n-2]}.$$

Each of these integrals, and the function $t \mapsto \mathrm{Vol}(X, \gamma(t))$, is a polynomial in t on some small interval $[0, t_0]$. As $\lim_{t \to 0} \mathrm{Vol}(X, \gamma(t)) > 0$ the function $t \mapsto h(t)$ is continuous and positive on a compact interval, so the integral $L(\gamma)$ of its square root exists and is finite. □

A holomorphic map $f : X \to Y$ between compact Kähler manifolds induces a morphism $f^* : \mathrm{H}^*(Y, \mathbb{R}) \to \mathrm{H}^*(X, \mathbb{R})$ in cohomology that respects the Hodge decomposition. However, if ω is a Kähler class on Y, then $f^*\omega$ is hardly ever a Kähler class on X. This happens mostly if f is either an embedding or a finite covering map.

Proposition 1.5. *Let $f : X \to Y$ be a finite surjective morphism. Let g_X and g_Y be the metrics on the Kähler cones of X and Y, respectively. Then the pullback morphism $f^* : \mathcal{C}(Y) \to \mathcal{C}(X)$ is a Riemannian embedding.*

Proof. Let ω be a point in $\mathcal{C}(Y)$. The volume of X with respect to $f^*\omega$ is

$$\mathrm{Vol}(X, f^*\omega) = p\ \mathrm{Vol}(Y, \omega)$$

as f is finite of degree p. It follows that f^* is an embedding. □

Corollary 1.6. *The group $\mathrm{Aut}\, X$ of holomorphic automorphisms of X acts by isometries on the Kähler cone $\mathcal{C}(X)$.*

A closer look reveals that this last statement contains less information than first meets the eye. The automorphism group $\mathrm{Aut}\, X$ of a compact complex manifold is a Lie group and it splits roughly into two parts; a positive-dimensional group given by the flows of holomorphic vector fields, or elements of $\mathrm{H}^0(X, T_X)$, and a discrete part consisting of "other" automorphisms. The isomorphisms generated by vector fields act trivially on the cohomology ring of X, so the only part of $\mathrm{Aut}\, X$ that possibly acts by nontrivial isometries on $\mathcal{C}(X)$ is discrete.

2. Connection and curvature

We start with a couple of preliminary computations.

Lemma 2.1. *If $u_1, \dots, u_k$ are real $(1,1)$-classes, then*

$$\begin{aligned} d_v \Lambda^{[k]}(u_1 \cup \dots \cup u_k) &= -\Lambda(v)\Lambda^{[k]}(u_1 \cup \dots \cup u_k) \\ &\quad + \Lambda^{[k]}(d_v u_1 \cup \dots \cup u_k) + \dots + \Lambda^{[k]}(u_1 \cup \dots \cup d_v u_k) \\ &\quad + \Lambda^{[k+1]}(u_1 \cup \dots \cup u_k \cup v). \end{aligned}$$

Proof. This is clear once we write

$$\Lambda^{[k]}(u_1 \cup \dots \cup u_k) = \frac{1}{\mathrm{Vol}(X, \omega)} \int_X u_1 \cup \dots \cup u_k \cup \omega^{[n-k]}$$

and compute. □

Lemma 2.2. *Let u, v, z be $(1,1)$-classes. Then*

$$\langle \Lambda(u \cup v), z \rangle = -\Lambda^{[3]}(u \cup v \cup z) + \Lambda^{[2]}(u \cup v)\Lambda(z).$$

Proof. First note that if z is a $(1,1)$-class, then $z = (z - \frac{1}{n}\Lambda(z)\omega) + \frac{1}{n}\Lambda(z)\omega$ is its primitive decomposition. Then

$$\begin{aligned} *(\omega \cup z) &= *(\omega \cup (z - \tfrac{1}{n}\Lambda(z)\omega)) + *\Big(\tfrac{2}{n}\Lambda(z)\omega^{[2]}\Big) \\ &= -(z - \tfrac{1}{n}\Lambda(z)\omega) \cup \omega^{[n-3]} + \tfrac{2}{n}\Lambda(z)\omega^{[n-2]} \\ &= -z \cup \omega^{[n-3]} + \tfrac{n-2}{n}\Lambda(z)\omega^{[n-2]} + \tfrac{2}{n}\Lambda(z)\omega^{[n-2]} \\ &= -z \cup \omega^{[n-3]} + \Lambda(z)\,\omega^{[n-2]}. \end{aligned}$$

We now get

$$\begin{aligned} \langle \Lambda(u \cup v), z\rangle\, \omega^{[n]} &= \Lambda(u \cup v) \cup (-z \cup \omega^{[n-2]} + \Lambda(z)\,\omega^{[n-1]}) \\ &= -\Lambda^{[2]}(\Lambda(u \cup v) \cup z)\,\omega^{[n]} + 2\Lambda^{[2]}(u \cup v)\Lambda(z)\,\omega^{[n]}, \end{aligned}$$

which proves the result. □

Recall that the Levi-Civita connection is the unique connection on the tangent bundle that's compatible with the metric and is torsion-free. That is, it satisfies

$$d\langle u, v\rangle = \langle \nabla u, v\rangle + \langle u, \nabla v\rangle, \quad \nabla_u v - \nabla_v u = [u, v]$$

for all sections u, v of the bundle.

Proposition 2.3. *The Levi-Civita connection of the Riemannian metric g on $\mathcal{C}$ is*

$$\nabla_z u = d_z u - \tfrac{1}{2}\Lambda(u)z - \tfrac{1}{2}\Lambda(z)u + \tfrac{1}{2}\Lambda(u \cup z).$$

Proof. The connection we've written down satisfies $\nabla_u v - \nabla_v u = [u, v]$ by inspection. We turn to its computation.

The metric is defined by

$$\langle u, v\rangle = \Lambda u\, \Lambda v - \Lambda^{[2]}(u \cup v).$$

Taking the derivative of this in the z direction gives

$$\begin{aligned} d_z\langle u, v\rangle = {} & -\Lambda(u)\Lambda(v)\Lambda(z) + \Lambda(d_z u)\Lambda(v) + \Lambda^{[2]}(u \cup z)\Lambda(v) \\ & - \Lambda(u)\Lambda(v)\Lambda(z) + \Lambda(u)\Lambda(d_z v) + \Lambda(u)\Lambda^{[2]}(v \cup z) \\ & + \Lambda(z)\Lambda^{[2]}(u \cup v) - \Lambda^{[2]}(d_z u \cup v) - \Lambda^{[2]}(u \cup d_z v) - \Lambda^{[3]}(u \cup v \cup z) \\ =: {} & \langle d_z u, v\rangle + \langle u, d_z v\rangle + A(u, v, z). \end{aligned}$$

We're going to write $A = \frac{1}{2}A + \frac{1}{2}A$ and try to write the first half as an inner product with v, and the second half as an inner product with u.

To that end, we note that

$$\begin{aligned} A(u, v, z) = {} & -\Lambda(u)\Lambda(v)\Lambda(z) + \Lambda^{[2]}(u \cup z)\Lambda(v) \\ & - \Lambda(u)\Lambda(v)\Lambda(z) + \Lambda(u)\Lambda^{[2]}(v \cup z) \\ & + \Lambda(z)\Lambda^{[2]}(u \cup v) - \Lambda^{[3]}(u \cup v \cup z) \\ = {} & -\Lambda(z)\langle u, v\rangle - \Lambda(u)\langle z, v\rangle + \Lambda^{[2]}(u \cup z)\Lambda(v) - \Lambda^{[3]}(u \cup v \cup z) \\ = {} & -\langle \Lambda(z)u, v\rangle - \langle \Lambda(u)z, v\rangle + \langle \Lambda(u \cup z), v\rangle \end{aligned}$$

by Lemma 2.2, which can indeed be written as an inner product with v. Since A is symmetric in u, v, z, we can start again from A and write it as an inner product with u. Taking half of each, we arrive at our claimed form of the connection. □

Corollary 2.4.

- $\nabla\omega = 0$.
- *If u is a primitive vector field, then ∇u is also primitive.*

Theorem 2.5. *The curvature tensor of the metric on the Kähler cone is*

$$R(u,v,z,w) = -\tfrac{1}{4}\langle\Lambda(u\cup w),\Lambda(v\cup z)\rangle + \tfrac{1}{4}\langle\Lambda(u\cup z),\Lambda(v\cup w)\rangle.$$

Proof. We may assume that all the tangent fields u, v, z, w are primitive, either by appealing to the isometric splitting of the Kähler cone, by using that $\nabla\omega = 0$ and the symmetries of the curvature tensor to see that R always degenerates to the primitive parts of our classes, or by calculating the curvature tensor first for primitive classes and then doing painful algebra to see that the general case degenerates to that one. However we do it, we find that for primitive fields we have

$$\nabla_v z = d_v z + \tfrac{1}{2}\Lambda(v\cup z)$$

and

$$\nabla_u\nabla_v z = d_u d_v z + \tfrac{1}{2}(d_u\Lambda)(v\cup z) + \tfrac{1}{2}\Lambda(d_u v\cup z) + \tfrac{1}{2}\Lambda(v\cup d_u z) + \tfrac{1}{2}\Lambda(u\cup d_v z) + \tfrac{1}{4}\Lambda(u\cup\Lambda(v\cup z)).$$

This gives

$$R(u,v)z = \tfrac{1}{2}(d_u\Lambda)(v\cup z) + \tfrac{1}{4}\Lambda(u\cup\Lambda(v\cup z)) - \tfrac{1}{2}(d_v\Lambda)(u\cup z) - \tfrac{1}{4}\Lambda(v\cup\Lambda(u\cup z))$$

since the other terms either make up $\nabla_{[u,v]}z$ or are symmetric in u, v. To make sense of this, it's convenient to take the inner product with w.

We have $*(\omega\cup w) = -\omega^{[n-3]}\cup w$ since w is primitive, so

$$\langle\Lambda(v\cup z), w\rangle = \langle v\cup z, \omega\cup w\rangle = -\Lambda^{[3]}(v\cup z\cup w).$$

Differentiating this in the direction of u gives

$$\langle(d_u\Lambda)(v\cup z), w\rangle + \tfrac{1}{2}\langle\Lambda(u\cup\Lambda(v\cup z)), w\rangle + \tfrac{1}{2}\langle\Lambda(v\cup z),\Lambda(u\cup w)\rangle = -\Lambda^{[4]}(u\cup v\cup z\cup w)$$

after canceling out the terms that involve the derivatives of the tangent fields. Then one part of the curvature tensor is

$$\tfrac{1}{2}\langle(d_u\Lambda)(v\cup z), w\rangle + \tfrac{1}{4}\langle\Lambda(u\cup\Lambda(v\cup z)), w\rangle = -\tfrac{1}{2}\Lambda^{[4]}(u\cup v\cup z\cup w) - \tfrac{1}{4}\langle\Lambda(v\cup z),\Lambda(u\cup w)\rangle.$$

The first term is symmetric in u, v, so we get

$$R(u,v,z,w) = -\tfrac{1}{4}\langle\Lambda(u\cup w),\Lambda(v\cup z)\rangle + \tfrac{1}{4}\langle\Lambda(u\cup z),\Lambda(v\cup w)\rangle$$

as promised. □

Remark. If x, y are $(2,2)$-classes, then

$$\Lambda^{[4]}(x \cup y) = \langle x, y \rangle - \langle \Lambda(x), \Lambda(y) \rangle + \langle \Lambda^{[2]}(x), \Lambda^{[2]}(y) \rangle;$$

see [Mag16]. An alternate expression for the curvature tensor is thus

$$R(u,v,z,w) = -\tfrac{1}{4}\langle u,w\rangle\langle v,z\rangle + \tfrac{1}{4}\langle u,z\rangle\langle v,w\rangle \\ - \tfrac{1}{4}\langle u \cup w, v \cup z\rangle + \tfrac{1}{4}\langle u \cup z, v \cup w\rangle,$$

so the curvature tensor is a perturbation of the curvature tensor of a space form of constant sectional curvature. Unfortunately there is no known way to control the perturbation terms in general; at least bounding them from above is impossible by Wilson and Trenner's example [TW11].

On algebraic curvature tensors. The expression for the curvature tensor suggests that we could investigate the operation $(u, v) \mapsto \frac{1}{2}\Lambda(u \cup v)$ to understand the curvature of the metric. This operation defines an algebra structure on $\mathrm{H}^{1,1}(X, \mathbb{R})$; as an algebra, it is commutative, non-associative and non-unital (if it had a unit, it would have to be a multiple of ω, which doesn't work). This algebra structure varies as the Kähler class ω varies.

This curvature tensor conforms to a form of algebraic curvature tensors that, as far as the author knows, have not received much attention. We gather here some trivialities about them, the first of which suggests this will not be a fertile line of investigation.

Proposition 2.6. *Let V be a real vector space, equipped with an inner product $\langle\ ,\ \rangle$, and an algebra structure $(x, y) \mapsto x \cdot y$. If the algebra structure is commutative, then*

$$R(x,y,z,w) := \langle x \cdot w, y \cdot z \rangle - \langle x \cdot z, y \cdot w \rangle$$

is an algebraic curvature tensor.

Proof. It is immediate that $R(y,x,z,w) = R(x,y,w,z) = -R(x,y,z,w)$. The commutativity entails that $R(z,w,x,y) = R(x,y,z,w)$, so R defines a symmetric bilinear form on $\bigwedge^2 V$. The commutativity also entails that R satisfies the Bianchi identity

$$R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0. \qquad \square$$

The moral of this proposition is perhaps that one should not expect to be able to prove very much about our curvature tensor from formal properties alone. After all, the commutative algebra structures on a vector space of dimension $h^{1,1}$ form a vector space of dimension $(h^{1,1})^2(h^{1,1}+1)/2$, while we can at best expect to generate a space of dimension $h^{1,1}$ therein by deforming our Kähler classes. Without any way of distinguishing the structures defined by a Kähler class (if any) from the others, and without this class of curvature tensors having some special properties, there is then little hope of progress in this direction.

Recall that if b is a symmetric bilinear form on V, then the Kulkarni–Nomizu product of b is the algebraic curvature tensor defined as

$$(b \wedge b)(x, y, z, w) = b(x, z)b(y, w) - b(x, w)b(y, z).$$

Proposition 2.7. *A curvature tensor defined by an algebra is a sum of Kulkarni–Nomizu products.*

Proof. Let $(x_1, \dots, x_n)$ be an orthonormal basis of V. Define bilinear forms $b_l(x, y) := \langle x \cdot y, x_l \rangle$. These are symmetric as the algebra is symmetric, and satisfy $x \cdot y = \sum_{l=1}^{n} b_l(x, y)x_l$. It follows that

$$R(x, y, z, w) = -\sum_{l=1}^{n} (b_l \wedge b_l)(x, y, z, w). \qquad \square$$

The curvature tensors of our algebra structures are made up of symmetric bilinear forms on S^2V. If x, y are vectors in V, we'll write xy for the induced vector $\frac{1}{2}(x \otimes y + y \otimes x)$ in S^2V. The inner product on V induces an inner product on S^2V by

$$\langle xy, zw \rangle = \tfrac{1}{2}\big(\langle x, z\rangle\langle y, w\rangle + \langle x, w\rangle\langle y, z\rangle\big).$$

We'll say that an algebraic curvature tensor has constant sectional curvature if it is equal to a multiple of the Kulkarni–Nomizu product of an inner product.

Proposition 2.8. *An algebraic curvature tensor induced by an algebra structure has constant sectional curvature if and only if there exists a scalar λ and a symmetric 4-tensor A such that*

$$\langle x \cdot y, z \cdot w \rangle = 2\lambda \langle xy, zw \rangle + A(x, y, z, w)$$

for all vectors $x, y, z, w \in V$.

Proof. If this condition holds, it is a simple computation to show that the curvature tensor defined by the algebra has constant sectional curvature $-\lambda$ with our sign choices.

Conversely, if the curvature tensor has constant sectional curvature $-\lambda$, the symmetries of the curvature tensor entail that the linear form

$$A(x, y, z, w) := \langle x \cdot w, y \cdot z \rangle - 2\lambda \langle xw, yz \rangle$$

is symmetric. $\square$

Recall that a derivation D of an algebra is a linear map on the underlying vector space such that $D(x \cdot y) = Dx \cdot y + x \cdot Dy$ for all vectors x, y. The derivations of an algebra on V form a subalgebra of $\operatorname{End} V$. One can constrain them a little by formal manipulations in the case of our algebra:

Proposition 2.9. *If D is a derivation of the algebra on $\mathrm{H}^{1,1}(X, \mathbb{R})$, then $D\omega = 0$, Dx is primitive for all x, and $D^t = -D$.*

Proof. We have $[L,\Lambda] = (2-n)\,\mathrm{id}$ on $(1,1)$-forms. We can interpret this as a statement about the algebra product of a class with ω:

$$x \cdot \omega = \tfrac{1}{2}\Lambda(x)\omega + \tfrac{1}{2}(n-2)x.$$

We have $\omega \cdot \omega = (n-1)\omega$. Then

$$(n-1)D\omega = 2D\omega \cdot \omega = \Lambda(D\omega)\,\omega + nD\omega.$$

Then

$$D\omega = -\Lambda(D\omega)\,\omega,$$

that is, $D\omega$ is a multiple of ω.

Suppose that u is primitive. Then $u \cdot \omega = \frac{1}{2}(n-2)u$. We get

$$\begin{aligned} \tfrac{1}{2}(n-2)Du &= Du \cdot \omega + u \cdot D\omega \\ &= \tfrac{1}{2}\Lambda(Du)\,\omega + \tfrac{1}{2}(n-2)Du + \Lambda(D\omega)\tfrac{1}{2}(n-2)u, \end{aligned}$$

That is,

$$\Lambda(Du)\,\omega + (n-2)\Lambda(D\omega)\,u = 0.$$

As ω and u are orthogonal, the only way this can hold is if $\Lambda(Du) = \Lambda(D\omega) = 0$. Then we also get $D\omega = 0$.

It follows that Dx is primitive for any $(1,1)$-class x. For primitive classes u and v, we have

$$D(u \cdot v) = Du \cdot v + u \cdot Dv.$$

Note that if either u or v is a primitive class, then $\Lambda(u \cdot v) = -\langle u, v\rangle$. From the above, it follows that

$$\langle Du, v\rangle + \langle u, Dv\rangle = 0,$$

so the linear morphism D satisfies $D^t = -D$. □

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